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Derivatives of Entire Functions of Higher Order

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Bounds for the derivative of entire functions of higher order are related to bounds on the functions themselves. S. N. Bernstein proved the following result for the derivative of entire functions of finite type: if f is an entire function of exponential type λ and $|f(x)| \leqslant M$ for $x \in \mathbb{R}$, then $|f'(x)| \leqslant \lambda M$ on $\mathbb{R}$. In Volume 58 of the *Journal of Approximation Theory*, R. A. Zalik raised the problem of extending Bernstein's estimate to entire functions of higher order. But he finally considered another question; namely, he proved: Let f be an entire function, $n \geqslant 0$, A_1, A_2, a, b, c, d real numbers and write $z = x + iy$. If $|f(z)| \leqslant (A_1 + A_2 |z|^n) \exp(ax^2 + by^2 + cx + dy)$ for all $z \in \mathbb{C}$, then there are numbers C_1, $C_2 \geqslant 0$ depending only on n, A_1, A_2, a, b, c, and d such that $|f'(z)| \leqslant (C_1 + C_2 |z|^{n+1}) \exp(ax^2 + by^2 + cx + dy)$.

In this note first I would like to point out that even the following more general result is an easy consequence of Cauchy's integral formula:

PROPOSITION 1. *Let $P(x, y)$ be a polynomial of two variables of degree k, $k \geqslant 1$, and $\alpha \geqslant 0$. If f is entire and*

$$|f(z)| \leqslant (1 + |z|^{\alpha}) \exp(P(x, y))$$

for all $z \in \mathbb{C}$, then with a constant C depending only on α and P we have

$$|f'(z)| \leqslant C(1 + |z|^{\alpha + k - 1}) \exp(P(x, y)).$$

For the proof it is enough to apply Cauchy's formula

$$f'(z) = \frac{1}{2\pi i} \oint_{|\zeta - z| = \tau} \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta \tag{1}$$

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on the derivative of f with $\tau = 1/(1+|z|)^{k-1}$ and note that for $|\zeta - z| \leqslant 1/(1+|z|)^{k-1}$, $\zeta = \xi + i\eta$, we have $P(\xi, \eta) \leqslant P(x, y) + C_1$ with a constant C_1 depending on P only.

It is somewhat more interesting that the above result has a certain converse. To state this we recall that P can be written as a sum

$$P(x, y) = R_k(x, y) + R_{k-1}(x, y) + \cdots + R_0, \tag{2}$$

where R_j is a homogeneous polynomial of degree j in two variables.

PROPOSITION 2. *If P and α are as in Proposition* 1 *and R_k (see* (2)) *is positive definite, then for an entire function f the condition*

$$|f'(z)| \leqslant (1 + |z|^{\alpha+k-1}) \exp(P(x, y))$$

implies

$$|f(z)| \leqslant C(1 + |z|^{\alpha}) \exp(P(x, y))$$

with a constant C depending only on P and α.

Let us immediately note that the conclusion is false if we merely assume that R_k is positive *semi*-definite even if the strict positive definiteness of P is assumed. In fact, for $f(z) = (1+z^3)\exp(z^2)$ we have the estimate

$$|f'(z)| \leqslant C(1 + |z|^4) \exp(y^4 + x^2 + y^2),$$

but $|f(z)| \leqslant C_1 \exp(y^4 + x^2 + y^2)$ is not satisfied for large positive z.

Proof of Proposition 2. The positive definiteness of R_k implies that for large $|z|$ in P the dominant term will be R_k, hence the following relations are easy to verify for large $|z|$, say for $|z| \geqslant M$,

$$P(x, y) \geqslant c\,|z|^k,$$

$$\frac{dP(\lambda x, \lambda y)}{d\lambda} \geqslant c\,|z|^k \qquad \text{for} \quad \tfrac{1}{2} \leqslant \lambda \leqslant 1,$$

$$\max_{0 \leqslant \lambda \leqslant 1/2} P(\lambda x, \lambda y) \leqslant P(\tfrac{1}{2}x, \tfrac{1}{2}y) \leqslant P(x, y) - k \log |z|,$$

where the constant $c > 0$ depends only on P. These imply for $|z| > M$,

$$\begin{aligned} |f(z/2) - f(0)| &= \left| z \int_0^{1/2} f'(\lambda z)\, d\lambda \right| \\ &\leqslant |z|\ (1 + |z|^{\alpha+k-1})\ \tfrac{1}{2} \max_{\lambda \leqslant 1/2} e^{P(\lambda x, \lambda y)} \\ &\leqslant C\,|z|^{\alpha}\, e^{P(x, y)} \end{aligned}$$

and

$$
\begin{aligned}
|f(z/2)-f(z)| &= \left| z \int_{1/2}^{1} f'(\lambda z)\, d\lambda \right| \\
&\leqslant |z|\, (1+|z|^{\alpha+k-1})(c\,|z|)^{-k} \left| \int_{1/2}^{1} e^{P(\lambda x, \lambda y)} \frac{dP(\lambda x, \lambda y)}{d\lambda}\, d\lambda \right| \\
&\leqslant C\, |z|^{\alpha}\, e^{P(x, y)},
\end{aligned}
$$

which prove the proposition. ∎

After these let us turn to the original question concerning bounds for the derivative on the real line of an entire function of higher order provided the function is bounded on $\mathbb{R}$.

THEOREM. *Suppose that f is an entire function such that for some constants $\alpha \geqslant 1$ and c, $C>0$ we have*

$$|f(z)| \leqslant Ce^{c|z|^{\alpha}}, \qquad z \in \mathbb{C}. \tag{3}$$

If f is bounded on the positive half-line, say,

$$|f(x)| \leqslant M, \qquad x \geqslant 0,$$

then for $x \geqslant 0$ we have

$$|f'(x)| \leqslant C_1 \max(C, M)(1+x^{\alpha-1})$$

with a constant C_1 depending only on α and c.

For integer α the function $f(z)=\sin(z^{\alpha})$ shows that this is in general the best possible estimate. We immediately note that the somewhat stronger statement

$$|f'(x)| \leqslant C_1(1+x^{\alpha-1})\, \|f\|_{\mathbb{R}} \tag{4}$$

with a constant C_1 depending only on α, c, and C, which would be the perfect analogue of Bernstein's result is not true: Consider the functions

$$f_T(z)=e^{-T^2}e^{-z^2}\sin Tz, \qquad T>0.$$

For all of these we can put $\alpha=c=C=2$ to have the estimate (3) with f replaced by f_T. But $\|f_T\|_{\mathbb{R}} \leqslant e^{-T^2}$, while $f'_T(0)=Te^{-T^2}$, so (4) is not satisfied if T is sufficiently large.

As a corollary we get the following more general result.

COROLLARY. *Suppose that f is an entire function such that for some constants $\alpha \geqslant 1$ and c, $C > 0$ condition* (3) *holds. If on the positive half-line, f satisfies*

$$|f(x)| \leqslant M(1+x^{\beta})e^{\delta x^{\gamma}}, \qquad x \geqslant 0,$$

with some constants $\beta \geqslant 0$, δ, and γ, then for $x \geqslant 0$, we have

$$|f'(x)| \leqslant C_1 \max(C, M)(1+x^{\alpha-1})(1+x^{\beta})e^{\delta x^{\gamma}}$$

with a constant C_1 depending only on α, c, β, δ, and γ.

Proof of the theorem. The proof will be similar to the Phragmén–Lindelöf arguments. Let us suppose first that α is an integer, and for $\varepsilon > 0$ consider the function

$$F(z) = f(z) \exp(i2cz^{\alpha} - \varepsilon z^{2\alpha})$$

in the sector $R_{\alpha} = \{z \mid 0 \leqslant \arg(z) \leqslant \pi/6\alpha\}$. Since for $\arg(z) = \pi/6\alpha$ we have $\Re(i2cz^{\alpha}) = -c\,|z|^{\alpha}$, we get that on the boundary of R_{α} the function F is bounded; more precisely,

$$|F(z)| \leqslant M \text{ if } \arg(z) = 0 \qquad \text{and} \qquad |F(z)| \leqslant C \text{ if } \arg(z) = \pi/6\alpha.$$

Because of the term $-\varepsilon z^{2\alpha}$ we also have $F(z) \to 0$ uniformly in $z \in R_{\alpha}$, $|z| \to \infty$; hence the maximum modulus principle implies that $|F(z)| \leqslant \max(C, M)$ if $z \in R_{\alpha}$; i.e., for $z \in R_{\alpha}$ the estimate $|f(z)| \leqslant \max(C, M)\,|\exp(-i2cz^{\alpha} + \varepsilon z^{2\alpha})|$ holds. If we let ε tend to zero we finally conclude

$$|f(z)| \leqslant \max(C, M)\,|\exp(-i2cz^{\alpha})|,$$

which implies for $x \geqslant 1$, $|z-x| \leqslant (1+x)^{-\alpha+1}$, $\Im z \geqslant 0$,

$$|f(z)| \leqslant C_1 \max(C, M)$$

with a constant C_1 depending only on α and c. A similar estimate follows for $x \geqslant 1$, $|z-x| \leqslant (1+x)^{-\alpha+1}$, $\Im z \leqslant 0$, and, on applying Cauchy's formula (1) on the circle $\{z \mid |z-x| = (1+x)^{-\alpha+1}\}$, we obtain

$$|f'(x)| \leqslant C_1 \max(C, M)(1+x^{\alpha-1}).$$

This proves the theorem for the case when α is an integer. For other α's repeat the above argument by taking that branch of z^{α} on the complex plane cut along the negative half-line which is positive for positive z. ∎

Proof of the corollary. Since (3) is true for f, we can assume without loss of generality that $\gamma \leqslant \alpha$. By taking appropriate branches (see the preceding proof) of the functions z^α, z^β, and z^γ, we get that the function

$$f^*(z) = f(z)(1+z^\beta)^{-1} \exp(-\delta z^\gamma)$$

is holomorphic in a sector $-\varphi \leqslant \arg(z) \leqslant \varphi$ for some $\varphi > 0$ and continuous on its boundary. Hence we can repeat the proof of the theorem by looking at a function

$$F^*(z) = f^*(z) \exp(iLz^\alpha - \varepsilon z^{2\alpha})$$

with some large L in the sector $-\varphi/2 \leqslant \arg(z) \leqslant \varphi/2$. We can again conclude

$$|f^{*\prime}(x)| \leqslant C_2 \max(C, M)(1 + x^{\alpha-1}),$$

from which the conclusion of the corollary follows for $x \geqslant 1$ by simple algebra. For $0 \leqslant x \leqslant 1$ the conclusion is a consequence of (3). ∎

References

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